(Motivic) tensor-triangular geometry

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Abstract

These are notes accompanying a series of 4 lectures at the *Motivic Midlands* workshop in Nottingham, 2026. They give a brief introduction to tensor-triangular geometry with examples drawn from algebraic geometry, and later on specifically from the motivic theory. Emphasis is put on general tt-geometric methods and how they apply in these examples. Only some familiarity with (derived) categorical language is assumed.

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1 Introduction

Consider the following two problems:

1. The *classification problem*: If C is your favourite category (arising in algebraic or arithmetic geometry, say), classifying its objects up to isomorphism is almost never possible. There are at least two ways to deal with this. One can restrict to subcategories of interest, or one can weaken the equivalence relation to be employed. For example, you won't be able to produce a manageable and useful list of all algebraic varieties up to isomorphism.

A typical move is then to study them up to birational equivalence. If one in addition restricts to curves, a useful classification can be given in terms of function fields.

2. The *categorification problem*: Often we study objects indirectly through their effect on other objects. For example, representation theory is about groups (or group-like objects) as viewed through their action on vector spaces (or other "linear" objects). More generally, rings are studied through their modules and algebraic varieties through sheaves on them. It is then natural to ask what these categories of sheaves remember about the varieties and how much information is lost. For example, a well-known result says that a variety X can be completely reconstructed from its category of quasi-coherent sheaves $C(X)$. This suggests a more ambitious approach, to wit, treating objects like $C(X)$ as if they were algebro-geometric objects themselves and studying them with algebro-geometric methods.

It doesn't seem obvious at all that these two problems should be related. However, we will see how tensor-triangular geometry can be seen as providing exactly such a link. In short, if the category C is "derived" (triangulated or a stable ∞-category) and has a "tensor structure" then there is a way to associate a space $\mathrm{Spec}(C)$, called the spectrum, which behaves in a similar way to the affine scheme $\mathrm{Spec}(R)$ associated with a commutative ring R. This is the starting point of doing algebraic geometry with such categories as input. Moreover, $\mathrm{Spec}(C)$ in a precise way classifies the objects of C up to a weaker equivalence relation, called tt-equivalence. Finally, if X is an algebraic variety and we take $C(X)$ to be the category of perfect complexes on X (the derived category of coherent sheaves if X is non-singular) then $\mathrm{Spec}(C(X))$ recovers X.

This theory has many precursors but was developed to a large extent by Paul Balmer, see e.g. [Bal10b]. The first goal of this mini-course is to explain in detail the preceding paragraph (that is, the spectrum, its relation to classification problems, and the reconstruction of X from $C(X)$ due to Thomason). In other words, the aim is to provide an introduction to tensor-triangular (or just tt-, from now on) geometry with examples drawn from algebraic geometry. One of the beautiful things about tt-geometry is that it applies in so many contexts since tt-categories occur so frequently (in algebraic geometry, representation theory, algebraic topology…). In the spirit of this workshop one can then ask what tt-geometry can tell us about the theory of motives. If we plug in a tt-category C of motives, what is $\mathrm{Spec}(C)$? Can we classify motives up to tt-equivalence? Thereby we enter the field of motivic tt-geometry. The second goal of this mini-course is to give a short overview of what is known in this direction.

I thank the organizers, Jean Paul Schemeil and Fraser Sparks, for the opportunity to present this material at their workshop.

2 Preliminaries

Remark 2.1. As mentioned, tt-geometry will apply to 'derived' contexts and we need to decide what language to use in dealing with these contexts (triangulated, stable ∞-, dg-, …). Before we do that let us stress that the outcome (at least as far as we will cover in these notes) does not depend on this decision (and it is straightforward to translate the constructions between the languages). I will let the audience decide on the language but in the notes we chose triangulated categories as being arguably the most accessible option (at least to the very little extent we will actually use them).

Definition 2.2. Recall that a *triangulated category* is an additive category C together with an autoequivalence $\Sigma\colon C \to C$ (called the *shift* or *suspension*) and a collection of sequences

$$a \xrightarrow{f} b \xrightarrow{g} c \to \Sigma a \tag{2.3}$$

called *distringuished triangles*, satisfying a list of axioms that suggest to think of c as the derived cokernel of f (we call it the *cone* or *cofiber*) and of a as the derived kernel of g (we call it *cocone* or *fiber*). Sometimes, b is called an *extension* of a and c (or of c by a if the order is important).

Remark 2.4. In particular, given such a distringuished triangle, applying $\mathrm{Hom}_C(x, -)$ gives rise to a long exact sequence of abelian groups.

Example 2.5. Let A be an abelian category and consider $\mathsf{Cpx}(A)$, the category of complexes in A. The localization $\mathsf{D}(A) := \mathsf{Cpx}(A)[\mathrm{qis}^{-1}]$ with respect to quasi-isomorphisms is the (unbounded) derived category of A and it has a[1] canonical structure of triangulated category. The shift is given by translating a complex to the left by one, and distinguished triangles arise from the mapping cone construction.

Example 2.6. Continuing with the previous example, let $M_1 \to M_2 \to M_3$ be a short exact sequence in A. There is a distinguished triangle in $\mathsf{D}(A)$ of the form $M_1 \to M_2 \to M_3 \to \Sigma M_1$. For any $L \in A$, applying $\mathrm{Hom}_{\mathsf{D}(A)}(L, -)$ yields the long exact sequence of Ext-groups, due to $\mathrm{Hom}_{\mathsf{D}(A)}(L, \Sigma^i M) = \mathrm{Ext}^i_A(L, M)$.

Definition 2.7. Let C be a triangulated category. A *triangulated subcategory* $C' \subseteq C$ is a full additive subcategory closed under $\Sigma^{\pm 1}$ and extensions. A *thick (triangulated) subcategory* is a triangulated subcategory that is closed under direct summands: if $a \oplus b \in C'$ then $a \in C'$.

Example 2.8. Given a collection of objects $\mathcal{X}$ in a triangulated category there is a smallest thick (resp. triangulated) subcategory containing $\mathcal{X}$. We denote it by $\mathrm{thick}(\mathcal{X})$.

Example 2.9. Let X be a scheme, by which we will always mean a quasi-compact quasi-separated scheme, for example a variety. There are various "derived categories" one associates with X. We will be interested in the (derived) category of *perfect complexes*, constructed as follows. Let $A = \mathrm{Mod}(\mathcal{O}_X)$ be the category of $\mathcal{O}_X$-modules and consider $\mathsf{D}(\mathrm{Mod}(\mathcal{O}_X))$ as above. The full subcategory of perfect complexes[2] is denoted $\mathsf{D}^{\mathrm{perf}}(X)$. It is a thick subcategory.

Remark 2.10. If $X = \mathrm{Spec}(R)$ is affine then $\mathsf{D}(\mathrm{Mod}(\mathcal{O}_X)) = \mathsf{D}(R)$ and $\mathsf{D}^{\mathrm{perf}}(X) = \mathrm{thick}(R)$ where R is viewed as a complex concentrated in degree 0.

If X is a noetherian regular scheme then $\mathsf{D}^{\mathrm{perf}}(X)$ is nothing but the bounded derived category of coherent sheaves on X.

[1] the pedant says: "actually two"...

[2] Recall that a complex of $\mathcal{O}_X$-modules is perfect if locally it is isomorphic to a bounded complex of vector bundles (=locally free sheaves of finite rank)

Exercise 2.11. Show that a triangulated subcategory has the 2-out-of-3 property for triangles: if two out of a, b, c in (2.3) belong to the subcategory then so does the third. (So, a triangulated subcategory is indeed canonically triangulated itself.)

Definition 2.12. An exact functor $F\colon C \to C'$ between triangulated categories is an additive functor $F\colon C \to C'$ together with a natural isomorphism $\Sigma' \circ F \cong F \circ \Sigma$ which sends distinguished triangles to distinguished triangles.

Example 2.13. A morphism $f\colon X \to Y$ of schemes induces, by pulling back, an exact functor $f^*\colon \mathsf{D}^{\mathrm{perf}}(Y) \to \mathsf{D}^{\mathrm{perf}}(X)$. (In fact, it does this at the level of $\mathsf{D}(\mathrm{Mod}(\mathcal{O}_Y)) \to \mathsf{D}(\mathrm{Mod}(\mathcal{O}_X))$ and then restricts to perfect complexes, essentially because vector bundles pull back to vector bundles.)

Exercise 2.14. An exact functor is an equivalence iff the underlying functor (forgetting the triangulated structure) is an equivalence.

The other 't' in tt-geometry stands for the tensor product, the abstract structure of which we recall below. The example to keep in mind, of course, is the usual tensor product of vector spaces or abelian groups, or more generally, R-modules when R is commutative.

Definition 2.15. A *symmetric monoidal category* is a category C and a bifunctor $\otimes\colon C \times C \to C$ (called the *tensor product*), a unit $\mathbb{1} \in C$ together with natural isomorphisms expressing associativity, symmetry and unitality. In principle there is a lot of structure and axioms to unpack. However, for our purposes these will play little to no role. The only thing to remember is that $a \otimes b \cong b \otimes a$, $\mathbb{1} \otimes a \cong a$ and that we may unambiguously write $a \otimes b \otimes c$ without parentheses.

Remark 2.16. Similarly, a *symmetric monoidal functor* is a functor between symmetric monoidal categories that preserves the tensor product and unit up to natural isomorphisms. Again, in principle there are lots of coherences such a functor is required to satisfy. However, for our purposes we can ignore these.

Example 2.17. The category $\mathrm{Mod}(\mathcal{O}_X)$ has a symmetric monoidal structure given by $- \otimes_{\mathcal{O}_X} -$, the tensor product over $\mathcal{O}_X$. The unit is $\mathcal{O}_X$.

Definition 2.18. A *tt-category* is a triangulated category C with a symmetric monoidal structure such that the tensor product is exact in each variable separately. A *tt-functor* is an exact symmetric monoidal functor between tt-categories.

Example 2.19. The derived category $\mathsf{D}(\mathrm{Mod}(\mathcal{O}_X))$ has the structure of a tt-category where the tensor product is left-derived from the tensor product $\otimes_{\mathcal{O}_X}$. For example, given two $\mathcal{O}_X$-modules M, N,

$$M \otimes^{\mathsf{L}}_{\mathcal{O}_X} N = F^M_\bullet \otimes_{\mathcal{O}_X} N = M \otimes_{\mathcal{O}_X} F^N_\bullet$$

for any flat resolutions of M or N, respectively.[3] This symmetric monoidal structure restricts to one on $\mathsf{D}^{\mathrm{perf}}(X)$, essentially because the tensor product of two vector bundles remains a

[3] In the sequel, functors will be assumed derived by default and we will omit making this explicit.

vector bundle and because the unit is still $\mathcal{O}_X$ which is perfect.

3 Spectrum

We now have the language to make the crucial definitions. All the material in this section is due to Paul Balmer.

3.1 Definition

Definition 3.1. Let C be a tt-category. An *ideal* $I \subseteq C$ is a thick subcategory closed under $C \otimes -$.

Remark 3.2. In the literature, these are also called "thick ideals" or "tt-ideals".

Remark 3.3. Obviously, the terminology is inspired by commutative algebra. Let us make the (rough) analogy explicit:

abelian group	triangulated category
addition	shifts, extensions, direct summands
multiplication	tensor product
1	$\mathbb{1}$
0	0
commutative ring	tt-category

Table 1: Categorification

This analogy will continue to guide us in this section.

Exercise 3.4. Show that ideals are precisely the kernels of tt-functors. This gives another justification for wanting to classify ideals (as we are about to do).

Exercise 3.5. 0 is the minimal ideal and C is the maximal ideal. The intersection of ideals is an ideal. In particular, given any collection of objects $X \subseteq C$ there is a smallest ideal containing it. It is denoted by $\langle X \rangle$. It consists of those objects that can be built out of objects in X in finitely many steps, using shifts, extensions, direct summands, and tensoring with objects in C.

We are now in a position to make precise the weaker equivalence relation alluded to in the introduction.

Definition 3.6. Let $a, b \in C$ be two objects in a tt-category. They are called *tt-equivalent* if $\langle a \rangle = \langle b \rangle$. In other words, a and b are tt-equivalent if they can be built out of each other in finitely many steps, as above.

Exercise 3.7. Let k be a field and consider $C = \mathsf{D}^{\mathrm{perf}}(k)$. Show that 0 and C are the only ideals. Hence any two non-zero objects are tt-equivalent.

The following definition is again analogous to the one in commutative algebra.

Definition 3.8. Let C be an essentially small tt-category.

1. A *prime ideal* is a proper ideal $P \subseteq C$ such that $a \otimes b \in P$ implies $a \in P$ or $b \in P$.
2. The spectrum $\mathrm{Spec}(C)$ is the set of prime ideals together with a basis for the topology given by
$$U(a) := \{P \in \mathrm{Spec}(C) \mid a \in P\}.$$
These are the distinguished opens. Their complements $\mathrm{supp}(a) := U(a)^{\mathsf{c}}$ are called *supports*.

Remark 3.9. Essential smallness makes $\mathrm{Spec}(C)$ into a (small) set. There are other, more subtle, reasons for focusing on (essentially) small tt-categories that we won't get into here.

Exercise 3.10. Check that this is well-defined. More generally, prove that

1. $\mathrm{supp}(\mathbb{1}) = \mathrm{Spec}(C)$, $\mathrm{supp}(0) = \emptyset$,
2. $\mathrm{supp}(a \otimes b) = \mathrm{supp}(a) \cap \mathrm{supp}(b)$
3. $\mathrm{supp}(a \oplus b) = \mathrm{supp}(a) \cup \mathrm{supp}(b)$
4. $\mathrm{supp}(\Sigma a) = \mathrm{supp}(a)$
5. if $a_1 \to a_2 \to a_3$ form part of a triangle then $\mathrm{supp}(a_i) \cup \mathrm{supp}(a_j)$ is independent of $i \neq j$

Exercise 3.11. Let $F\colon C \to C'$ be a tt-functor. Show that $\mathrm{Spec}(F)\colon \mathrm{Spec}(C') \to \mathrm{Spec}(C)$ sending P to $F^{-1}(P)$ is well-defined and continuous. In fact, $\mathrm{Spec}(F)^{-1}(U(a)) = U(Fa)$. This extends to a contravariant functor from (essentially small) tt-categories to spectral spaces.

Remark 3.12. At this point you might expect a discussion of the structure sheaf on $\mathrm{Spec}(C)$. This is possible to define as long as C comes with an enhancement (stable ∞-categories or dg categories). And it would be an important ingredient in addressing the categorification problem mentioned in the introduction. However, in this mini-course we will not go in this direction and therefore won't have any need for the structure sheaf. See Remark 3.28 though.

Example 3.13. Let k be a field and consider $C = \mathsf{D}^{\mathrm{perf}}(k)$ as in Exercise 3.7. 0 is the only prime ideal so that $\mathrm{Spec}(C)$ is a singleton.

You will observe that $\mathrm{Spec}(k)$ happens to be a singleton as well. Indeed, in Corollary 4.15 will see that this is not a coincidence.

Exercise 3.14. Let $P, Q \in \mathrm{Spec}(C)$. We say that P specializes to Q and write $P \rightsquigarrow Q$ if $Q \in \overline{\{P\}}$. Show that this is equivalent to $Q \subseteq P$. (This is not a typo. It is indeed the opposite of what you might expect from commutative algebra: minimal ideals are closed while generic points are maximal ideals. See Remark 3.19 for an explanation.)

3.2 Universal property

Next we want to give an account (in fact, two accounts) of what the space $\mathrm{Spec}(C)$ measures.

Remark 3.15.
1. First, we observe that $\mathrm{Spec}(C)$ is a *spectral space*, that is, it is homeomorphic to the spectrum of a commutative ring. Equivalently, it is sober, quasi-compact opens are closed under finite intersections (even empty ones) and form a basis.
2. Recall that in commutative algebra the closed subsets correspond to radical ideals. Similarly, for $P \in \mathrm{Spec}(C)$ and $a \in C$ the condition $a \in P$ is equivalent to $a^{\otimes n} \in P$ if $n > 0$. Consequently $\mathrm{supp}(a) = \mathrm{supp}(a^{\otimes n})$ and we see that $\mathrm{Spec}(C)$ will not distinguish between ideals and their radical closure. (An ideal $I \subseteq C$ is called *radical* if $x^{\otimes n} \in I$ implies $x \in I$.)
3. The only surprise is that ideals do not correspond to closed subsets in $\mathrm{Spec}(C)$ but rather in its 'Hochster dual' $\mathrm{Spec}(C)^\vee$. This is the spectral space with the same underlying set and topology of opens generated by subsets that are closed and have a quasi-compact complement in $\mathrm{Spec}(C)$.

Proposition 3.16. *The association*

$$\{\, \textit{radical ideals in } C \,\} \leftrightarrow \{\, \textit{dual-open subsets of } \mathrm{Spec}(C) \,\}$$
$$I \mapsto \bigcup_{a \in I} \mathrm{supp}(a)$$

is an inclusion-preserving bijection. The inverse is given by sending $V \subseteq \mathrm{Spec}(C)$ to the set I_V of $a \in C$ with $\mathrm{supp}(a) \subseteq V$.

Exercise 3.17. You might find it illuminating to read up on Stone duality that establishes an antiequivalence between bounded distributive lattices and spectral spaces. For then you will see that f.g. radical ideals in a commutative ring as well as in an essentially small tt-category form such a bounded distributive lattice. In the former case the spectrum implements the equivalence above. In the latter case, the spectrum is the corresponding spectral space but with the Hochster-dual topology. Proposition 3.16 is then a formal consequence of this.

Exercise 3.18. As a formal consequence of Proposition 3.16 show that every radical ideal is the intersection of prime ideals.

Remark 3.19. You might wonder why $\mathrm{Spec}(C)$ has the Hochster-dual topology. At some level this is simply a convention: taking the Hochster-dual is an involution so that no information is lost. However, we will see later that geometric intuition demands the convention chosen in Definition 3.8. It makes functors that we think of as 'open' actually induce open maps on spectra, for example. In a similar vein, we will see later that with our convention $\mathrm{Spec}(\mathsf{D}^{\mathrm{perf}}(X)) = X$ (and not its Hochster dual).

Exercise 3.20. Let $V \subseteq X$ be a dual-open subset corresponding to the radical ideal $I \subseteq C$. What is the map on spectra induced by the canonical quotient $C \to C/I$?

Recall the basic properties 1.–5. in Exercise 3.10 that the supports satisfy. We turn this into a definition as follows.

Definition 3.21. Let T be a topological space and σ a function that associates to every object $c \in C$ a closed subset $\sigma(c) \subseteq T$. The pair (T, σ) is called a *support datum* on C if σ satisfies the analogous conditions of Exercise 3.10.

We can also make this into a category by having a morphism $f\colon (T', f^{-1} \circ \sigma) \to (T, \sigma)$ for every continuous map $f\colon T' \to T$ (and obvious composition).

The following result provides a universal property of the spectrum.

Proposition 3.22. *The pair* $(\mathrm{Spec}(C), \mathrm{supp})$ *is the final support datum on* C.

Explicitly, given a support datum (T, σ) *on* C*, the unique map to* $(\mathrm{Spec}(C), \mathrm{supp})$ *is defined by*

$$T \ni t \mapsto \{c \in C \mid t \notin \sigma(c)\}.$$

Exercise 3.23. Prove the proposition.

3.3 Categorification problem

Example 3.24. Let X be a (qcqs) scheme. Given a perfect complex $K \in \mathsf{D}^{\mathrm{perf}}(X)$, its cohomological support $\mathrm{Supp}^{\mathrm{h}}(K) \subseteq X$ is the support (in the algebraic-geometry sense) of its total cohomology $\oplus_i \mathrm{H}^i(K)$. Working affine locally on $\mathrm{Spec}(R)$, this total cohomology is a finitely generated R-module M and its support is the vanishing set of $\mathrm{Ann}_R(M)$. In particular, we see that $\mathrm{Supp}^{\mathrm{h}}(K) \subseteq X$ is a closed subset. In fact, you can check that this defines a support datum on $\mathsf{D}^{\mathrm{perf}}(X)$ so that there is a unique continuous map

$$\tau\colon X \to \mathrm{Spec}(\mathsf{D}^{\mathrm{perf}}(X)) \tag{3.25}$$

such that $\mathrm{Supp}^{\mathrm{h}}(K) = \tau^{-1}(\mathrm{supp}(K))$ for all perfect complexes K. Explicitly,

$$\tau(x) = \{K \in \mathsf{D}^{\mathrm{perf}}(X) \mid K_x = 0\}. \tag{3.26}$$

Thomason understood that the space X classifies ideals in $\mathsf{D}^{\mathrm{perf}}(X)$:

Theorem 3.27 ([Tho97]). *The map* (3.25) *is a homeomorphism.*

Proof. Our goal here is to reduce to the affine case which will be treated in the next section. For now, let $j\colon U \hookrightarrow X$ be any quasi-compact open subset, with complement Z. The restriction of perfect complexes $j^*\colon \mathsf{D}^{\mathrm{perf}}(X) \to \mathsf{D}^{\mathrm{perf}}(U)$ has kernel $\mathsf{D}^{\mathrm{perf}}_Z(X)$ those complexes K with $\mathrm{Supp}^{\mathrm{h}}(K) \subseteq Z$ by definition. The resulting functor

$$\mathsf{D}^{\mathrm{perf}}(X)/\mathsf{D}^{\mathrm{perf}}_Z(X) \to \mathsf{D}^{\mathrm{perf}}(U)$$

is in fact fully faithful and the image is characterized by the image of $K_0(X) \to K_0(U)$ [TT90]. It is easy to see that this functor induces a homeomorphism on spectra. (This is true more generally for fully faithful functors $D \to D'$ with "dense" image. This is a good exercise, using the fact that for any $d \in D'$ one has $d \oplus \Sigma d \in D$.) If you did Exercise 3.20 you'll then know that $\mathrm{Spec}(j^*)$ is the inclusion of the subset $\mathrm{supp}(\mathsf{D}^{\mathrm{perf}}_Z(X))^{\complement}$ into $\mathrm{Spec}(\mathsf{D}^{\mathrm{perf}}(X))$.

Claim: Let $(U_i)_i$ be a cover of X by quasi-compact opens, with complements Z_i. Then $(\mathrm{supp}(\mathsf{D}^{\mathrm{perf}}_{Z_i}(X))^{\complement})_i$ is a cover of $\mathrm{Spec}(\mathsf{D}^{\mathrm{perf}}(X))$. This is equivalent to the intersection of the $\mathrm{supp}(\mathsf{D}^{\mathrm{perf}}_{Z_i}(X))$ being empty, or also equivalently, any prime $\mathscr{P}$ containing at least one of the $\mathsf{D}^{\mathrm{perf}}_{Z_i}(X)$. But clearly,

$$\cap_i \mathsf{D}^{\mathrm{perf}}_{Z_i}(X) = 0 \subseteq \mathscr{P}$$

and hence by primality $\mathsf{D}^{\mathrm{perf}}_{Z_i}(X) \subseteq \mathscr{P}$ for some i. This proves the claim.

From the claim it follows that the statement of the theorem is local on X hence we may assume X is affine.[4] This is Corollary 4.15. □

Remark 3.28. Let C be a tt-category and $U \subseteq \mathrm{Spec}(C)$ a quasi-compact open, with complement corresponding to the radical ideal $I_{U^\complement} \subseteq C$. The ring $\mathrm{Hom}_{C/I_{U^\complement}}(\mathbb{1}, \mathbb{1})$ is commutative and contravariantly functorial in U. In other words, it defines a presheaf of commutative rings on the quasi-compact opens of $\mathrm{Spec}(C)$. Let $\mathscr{O}_C$ be its sheafification on $\mathrm{Spec}(C)$. It is called the *structure sheaf* (of rings) associated with C. Here are a few things you can check:

1. $(\mathrm{Spec}(C), \mathscr{O}_C)$ is a locally-ringed space.
2. For $C = \mathsf{D}^{\mathrm{perf}}(X)$ it recovers exactly the scheme X. (For this you want to use the result mentioned above that computes the quotient category $\mathsf{D}^{\mathrm{perf}}(X)/\mathsf{D}^{\mathrm{perf}}_{U^\complement}(X)$ as $\mathsf{D}^{\mathrm{perf}}(U)$ up to idempotent completion.)

Remark 3.29. The last point is worth emphasizing, for the following reason. Let X be a smooth projective k-variety. It is well-known that $\mathsf{D}^{\mathrm{b}}(\mathrm{Coh}(X)) = \mathsf{D}^{\mathrm{perf}}(X)$ is not a complete invariant of X *when viewed as a triangulated (or stable ∞-) category*. For example, for X an abelian variety, Mukai showed that $\mathsf{D}^{\mathrm{b}}(\mathrm{Coh}(X))$ and $\mathsf{D}^{\mathrm{b}}(\mathrm{Coh}(X^\vee))$ are equivalent [Muk81] as such. So, with regards to the categorification problem described in the introduction, we see that the tensor product is a crucial additional structure. (But see [BO01] for some cases where the tensor product is not required.)

Once one does use the tensor product and works at the level of symmetric monoidal stable ∞-categories, the structure sheaf above can be enhanced to a structure sheaf $\mathbb{O}$ of such categories, and the functor

$$X \mapsto (\mathrm{Spec}(\mathsf{D}^{\mathrm{perf}}(X)), \mathbb{O}_{\mathsf{D}^{\mathrm{perf}}(X)})$$

becomes a fully faithful functor from (qcqs) schemes to "categorified" schemes. We refer to [Aok+25] for such a perspective on the categorification problem.

4 Classification problem

In the introduction we claimed that tt-geometry had something to say not only about the categorification problem but also the classification problem. In this section we'll see how.

[4] This also uses naturality of τ, namely $\tau_X \circ j = (- \cap \mathrm{Spec}(\mathsf{D}^{\mathrm{perf}}(U))) \circ \tau_U$. This in turn follows easily from the explicit description of τ in (3.26).

Let C be a tt-category. On the one hand we called two objects $a, b \in C$ tt-equivalent if $\langle a\rangle = \langle b\rangle$ (Definition 3.6). On the other hand, we saw in the previous section that the space $\mathrm{Spec}(C)$ classifies radical ideals (Proposition 3.16). To be more precise, in this section we will explore a condition, rigidity, that forces all ideals to be radical. Under this condition we can then say that $\mathrm{Spec}(C)$ classifies tt-equivalence classes.

4.1 Rigidity

We have made the point that the tensor product (often) plays an important role in the story. We now single out a property, rigidity, that tends to make tt-geometry behave "better". We also point out that many examples in algebraic geometry, topology and representation theory are rigid (although not all of them).

Definition 4.1. An object $a \in C$ in a symmetric monoidal category is called *(strongly) dualizable* if there is $b \in C$ and morphisms $\mathbb{1} \to a \otimes b$ and $b \otimes a \to \mathbb{1}$ such that

$$a \to a \otimes b \otimes a \to a, \qquad b \to b \otimes a \otimes b \to b$$

are both the identity maps. If such a b exists it is uniquely determined, denoted by a^*, and called the *dual* of a. If all objects in C are dualizable then C is called *rigid*.

Exercise 4.2. Let R be a commutative ring. An R-module is dualizable iff it is finitely generated projective.

Example 4.3. More generally, for X a scheme, a complex K of $\mathcal{O}_X$-modules is dualizable in $\mathsf{D}(X)$ iff it is perfect. Indeed, K is dualizable iff the canonical map $\underline{\mathrm{Hom}}(K, \mathbb{1}) \otimes K \to \underline{\mathrm{Hom}}(K, K)$ is an equivalence. This can be checked locally, just as perfectness, and thereby one reduces to $X = \mathrm{Spec}(R)$ affine. It follows from the previous example and the following exercise that every perfect object is dualizable. The converse can be shown using that perfect objects are exactly the compact objects in $\mathsf{D}(R)$. Indeed, $\mathrm{Hom}(K, -) = \mathrm{Hom}(R, -) \circ \underline{\mathrm{Hom}}(K, -) \colon \mathsf{D}(R) \to \mathrm{Mod}(\mathbb{Z})$ and both functors on the right preserve coproducts: the first one because $\underline{\mathrm{Hom}}(K, -)$ is left adjoint to $- \otimes K$, the second one because $\mathrm{Hom}(R, -) = \mathrm{H}_0(-)$ and taking the direct sum of complexes is an exact operation.

Exercise 4.4. If C is a tt-category show that the dualizable objects form a tt-subcategory. (Without further assumptions on C this is probably quite tedious.)

Lemma 4.5. *Let $a \in C$ be dualizable. Then $\langle a\rangle = \langle a^{\otimes n}\rangle$ for any $n \geq 1$.*

Proof. It is clear that $\langle a^{\otimes n}\rangle \subseteq \langle a\rangle$. For the converse, the defining property of a dualizable object shows that a is a direct summand of $a^* \otimes a^{\otimes 2}$. In particular, $a \in \langle a^{\otimes 2}\rangle$ and so by induction $\langle a\rangle \subseteq \langle a^{\otimes n}\rangle$ as well. □

Corollary 4.6. *If C is a rigid tt-category then every ideal is radical. In particular,*

- *the ideals in C are in bijection with dual-open subsets of* $\mathrm{Spec}(C)$*;*
- $\langle a\rangle = \langle b\rangle$ *iff* $U(a) = U(b)$ *iff* $\mathrm{supp}(a) = \mathrm{supp}(b)$.

Remark 4.7. In particular, at least if C is rigid, the space $\mathrm{Spec}(C)$ can be said to classify tt-equivalence classes of objects in C. More precisely, there is a bijection (induced by taking the support)

$$\{\text{ objects in } C \;\} \;/\; \text{tt-equivalence} \leftrightarrow \\ \{\text{ closed subsets of } \mathrm{Spec}(C) \text{ with quasi-compact complement }\}$$

4.2 Decategorification and surjectivity

Here we discuss two general tools that are often useful (and will be useful to us in the sequel) in order to compute the spectrum. These were introduced in [Bal10a; Bal18], respectively.

For the first one note that the endomorphism ring of the unit in a tt-category C forms a graded-commutative ring $R_C := \mathrm{Hom}_C(\Sigma^\bullet \mathbb{1}, \mathbb{1})$. We denote by $\mathrm{Spec}(R_C)$ its spectrum, i.e. the set of graded prime ideals with the Zariski topology.

Definition 4.8. The association

$$\begin{aligned} \mathrm{comp}\colon \mathrm{Spec}(C) &\to \mathrm{Spec}(R_C) \\ \mathcal{P} &\mapsto \langle f \text{ homogeneous} \mid \mathrm{cone}(f) \notin \mathcal{P} \rangle \end{aligned}$$

defines a continuous map that is called the *comparison map*.

We may think of this as a form of decategorification just as viewing C as a ring object was a form of categorification.

Exercise 4.9. Show that for any morphism $f\colon \Sigma^n \mathbb{1} \to \mathbb{1}$ in C, $\mathrm{comp}^{-1}(Z(f)) = \mathrm{supp}(\mathrm{cone}(f))$ where $Z(f) \subseteq \mathrm{Spec}(R_C)$ is the vanishing locus of f.

Proposition 4.10 ([Bal10a, Theorem 7.3], [Lau23, Proposition 2.7])**.** *Assume C is rigid (e.g. generated by the unit (as a thick subcategory)) and R_C is noetherian. If* $\mathrm{comp}\colon \mathrm{Spec}(C) \to \mathrm{Spec}(R_C)$ *is injective then it is a homeomorphism.*

We now relate a criterion for a map $F\colon C \to C'$ to induce a surjection on spectra. For this we say that F is $\otimes$-*nilfaithful* if $F(f) = 0$ implies that f is $\otimes$-nilpotent, that is, $f^{\otimes n} = 0$ for some $n > 0$.

Proposition 4.11 ([Bal18, Theorem 1.3])**.** *Let C be rigid and assume that $F\colon C \to C'$ is $\otimes$-nilfaithful. Then* $\mathrm{Spec}(F)\colon \mathrm{Spec}(C') \to \mathrm{Spec}(C)$ *is surjective.*

Remark 4.12. Recall that a flat map of commutative rings induces a surjection on spectra if it is *faithfully* flat. $\otimes$-nilfaithfulness can be seen as an analogue of that condition.

Exercise 4.13. Let $f\colon a \to b$ in C with a dualizable. Consider the corresponding map $f'\colon \mathbb{1} \to b \otimes a^*$ obtained as the composition

$$\mathbb{1} \to a \otimes a^* \xrightarrow{f} b \otimes a^*.$$

Show:

1. $F(f) = 0$ iff $F(f') = 0$;
2. f is $\otimes$-nilpotent iff f' is.

As a consequence, in the criterion of Proposition 4.11 on may always assume the morphism to be of the form $f\colon \mathbb{1} \to b$.

4.3 (Graded) rings

Let R be a commutative dg-ring and $\mathsf{D}(R)$ its derived category. We denote by $\mathsf{D}^{\mathrm{perf}}(R)$ the thick subcategory generated by R. It is a tt-category with the symmetric monoidal structure $\otimes_R$.

The following result is a generalization of a famous computation due to Michael Hopkins [Hop87] and Amnon Neeman [Nee92]. (They treat the case where R is an ordinary (noetherian) commutative ring.) It is well-known, see e.g. [CI15, § 3].

Theorem 4.14. *Assume that R is formal (as a cdga). Then the comparison map*

$$\mathrm{comp}\colon \operatorname{Spec}(\mathsf{D}^{\mathrm{perf}}(R)) \to \operatorname{Spec}(\mathrm{H}_*(R))$$

is a homeomorphism.

Proof. 1. By assumption we may assume R has zero differentials. (So, it is just a graded-commutative ring.)

2. A filtered colimit of graded-commutative rings $R^{(i)}$ produces a limit

$$\varprojlim_i \operatorname{Spec}(\mathsf{D}^{\mathrm{perf}}(R^{(i)})) \xrightarrow{\varprojlim_i \mathrm{comp}} \varprojlim_i \operatorname{Spec}(R^{(i)})$$

in spaces. (This is clear for the target and not too difficult for the domain either. Or see [Gal18, Corollary 8.7] for a similar argument.) Hence we may assume R is noetherian of finite Krull dimension.

3. By Proposition 4.10, we only need to show injectivity.

4. Let $f \in R$ be a homogeneous nilpotent element. We claim that the base-change functor $\mathsf{D}^{\mathrm{perf}}(R) \to \mathsf{D}^{\mathrm{perf}}(R/f)$ is $\otimes$-nilfaithful. Indeed, given $\alpha\colon R \to M$ vanishing under this functor. This implies that the composite $R \xrightarrow{\alpha} M \to M \otimes \mathrm{cone}(f)$ also vanishes and therefore α factors through $M \xrightarrow{f} M$. But then $\alpha^{\otimes n}$ factors through $f^{\otimes n} = f^n$ which is zero for $n \gg 0$.

   It follows from Proposition 4.11 that the left vertical arrow in the following commutative square is surjective. (The right vertical arrow is a homeomorphism.)

$$\begin{tikzcd}
\operatorname{Spec}(\mathsf{D}^{\mathrm{perf}}(R/f)) \arrow[r, "\mathrm{comp}"] \arrow[d] & \operatorname{Spec}(R/f) \arrow[d] \\
\operatorname{Spec}(\mathsf{D}^{\mathrm{perf}}(R)) \arrow[r, "\mathrm{comp}"] & \operatorname{Spec}(R)
\end{tikzcd}$$

In particular, we may assume that R is reduced.

5. Let $\mathscr{P}, \mathscr{Q} \in \operatorname{Spec}(\mathsf{D}^{\mathrm{perf}}(R))$ mapping to the same prime ideal $\mathfrak{m} \in \operatorname{Spec}(R)$. The localization $R \to R_{\mathfrak{m}}$ induces a square

$$\begin{tikzcd} \operatorname{Spec}(\mathsf{D}^{\mathrm{perf}}(R_{\mathfrak{m}})) \arrow[r, "\mathrm{comp}"] \arrow[d] & \operatorname{Spec}(R_{\mathfrak{m}}) \arrow[d] \\ \operatorname{Spec}(\mathsf{D}^{\mathrm{perf}}(R)) \arrow[r, "\mathrm{comp}"] & \operatorname{Spec}(R) \end{tikzcd}$$

which is cartesian. Hence we may assume R to be a local ring and $\mathscr{P}, \mathscr{Q}$ mapping to the unique maximal graded ideal $\mathfrak{m}$.

6. We now prove the theorem by induction on the Krull dimension of R. If zero then R is a graded field (by the last two points) in which case every dg-module is free and the claim is clear.

7. If the Krull dimension is positive there is $f \in \mathfrak{m}$ a non-zero divisor. A similar argument as in step 4 shows that $\mathsf{D}^{\mathrm{perf}}(R) \to \mathsf{D}^{\mathrm{perf}}(R/f) \times \mathsf{D}^{\mathrm{perf}}(R[1/f])$ is $\otimes$-nilfaithful. But the square

$$\begin{tikzcd} \operatorname{Spec}(\mathsf{D}^{\mathrm{perf}}(R[1/f])) \arrow[r, "\mathrm{comp}"] \arrow[d] & \operatorname{Spec}(R[1/f]) \arrow[d] \\ \operatorname{Spec}(\mathsf{D}^{\mathrm{perf}}(R)) \arrow[r, "\mathrm{comp}"] & \operatorname{Spec}(R) \end{tikzcd}$$

is cartesian so that both $\mathscr{P}, \mathscr{Q}$ must come from R/f. We can then apply the induction hypothesis to conclude. □

Corollary 4.15. *Let R be a commutative ring. Then the comparison map* $\operatorname{Spec}(\mathsf{D}^{\mathrm{perf}}(R)) \to \operatorname{Spec}(R)$ *is a homeomorphism.*

Remark 4.16. Under the homeomorphism of Theorem 4.14, the support (in the sense of tt-geometry) of an object $M \in \mathsf{D}^{\mathrm{perf}}(R)$ corresponds to the usual support:

$$\mathrm{comp}(\mathrm{supp}(M)) = \mathrm{Supp}^{\mathrm{h}}(M) := \{\mathfrak{p} \mid M_{\mathfrak{p}} \neq 0\}$$

5 Motives

In another mini-course at this workshop you've been introduced to Voevodsky's category of geometric motives over a field $\mathbb{F}$, that we denote by $\mathsf{DM}(\mathbb{F})$, omitting a decoration for "geometric". It encodes a tremendous amount of arithmetic and geometric information, for example about the K-theory of $\mathbb{F}$ or algebraic cycles on $\mathbb{F}$-varieties. In some sense it does (or at least should) encode *all* information about $\mathbb{F}$-varieties that is of cohomological nature, and nothing else. In particular, classifying its objects up to isomorphism is impossible, and even more coarse-grained results about $\mathsf{DM}(\mathbb{F})$ are hard to come by.

It is also a rigid tt-category and one can ask questions like:

- Can we classify its objects up to tt-equivalence? That is, can we compute $\mathrm{Spec}(\mathsf{DM}(\mathbb{F}))$?
- Would the result tell us anything interesting?

This is the subject of *motivic tt-geometry*, and the short answers are No, and Yes.[5]

We start with a general observation. The endomorphism ring of the unit $\mathbb{1} = \mathrm{Spec}(\mathbb{F}) \in \mathsf{DM}(\mathbb{F})$ is $\mathbb{Z}$ and therefore the technology reported in Section 4.2 yields a comparison map

$$\mathrm{comp}\colon \mathrm{Spec}(\mathsf{DM}(\mathbb{F})) \to \mathrm{Spec}(\mathbb{Z}). \tag{5.1}$$

Proposition 5.2. *Let $\mathfrak{p} \in \mathrm{Spec}(\mathbb{Z})$ be a prime. The fiber of (5.1) over $\mathfrak{p}$ is*

1. $\mathrm{Spec}(\mathsf{DM}(\mathbb{F};\mathbb{Q}))$ *if* $\mathfrak{p} = 0$;
2. *the continuous image of* $\mathrm{Spec}(\mathsf{DM}(\mathbb{F};\mathbb{Z}/p))$ *if* $\mathfrak{p} = (p)$.

Proof. Since $\mathrm{Spec}(\mathbb{Q}) \subseteq \mathrm{Spec}(\mathbb{Z})$ results from inverting all $n \in \mathbb{Z}\backslash\{0\}$, it follows that the fiber over $\mathfrak{p} = 0$ results from inverting $n\colon \mathbb{1} \to \mathbb{1}$ for the same n, see [Bal10a, Corollary 5.6(c)]. But this is exactly $\mathsf{DM}(\mathbb{F};\mathbb{Q})$.

The scalar extension functor $\gamma_p\colon \mathsf{DM}(\mathbb{F}) \to \mathsf{DM}(\mathbb{F};\mathbb{Z}/p)$ admits a right adjoint that sends the unit to $\mathbb{Z}/p \in \mathsf{DM}(\mathbb{F})$. It follows from general tt-geometry [Bal18, Theorem 1.7] that the image of $\mathrm{Spec}(\gamma_p)$ is the support of $\mathbb{Z}/p$. But that subset is exactly the fiber of (5.1) over (p), by Exercise 4.9. □

The upshot is that to understand $\mathrm{Spec}(\mathsf{DM}(\mathbb{F}))$ (at least as a set) it is essentially sufficient to understand the rational and the finite-coefficient cases. These two turn out to exhibit very different behaviour.

Convention 5.3. For simplicity we will from now on assume that $\mathbb{F}$ is of characteristic zero.

5.1 Rational coefficients

The category $\mathsf{DM}(\mathbb{F};\mathbb{Q})$ is subject to Grothendieck's standard conjectures which posit the existence of certain algebraic cycles. These conjectures have been open for a long time and are considered to be very difficult. Even more difficult (in the precise sense that it implies the former[6], see [Bei12]) is the conjecture that

$$\mathsf{DM}(\mathbb{F};\mathbb{Q}) \simeq \mathsf{D}^{\mathrm{b}}(\mathsf{MM}(\mathbb{F})) \tag{5.4}$$

where $\mathsf{MM}(\mathbb{F})$ is the *category of mixed motives* over $\mathbb{F}$. It is an abelian category with an exact tensor product that makes it into a Tannakian category. A candidate for $\mathsf{MM}(\mathbb{F})$ and for the equivalence $\mathsf{DM}(\mathbb{F};\mathbb{Q}) \to \mathsf{D}^{\mathrm{b}}(\mathsf{MM}(\mathbb{F}))$ has been constructed by Nori [Nor], and independently by Ayoub [Ayo14] and Iwanari [Iwa14]. All these candidates are equivalent by [CG17].

The category $\mathsf{MM}(\mathbb{F})$ ties back to Grothendieck in that it is supposed to be the world of universal cohomological invariants of $\mathbb{F}$-varieties. The category of Chow motives over $\mathbb{F}$ modulo homological equivalence should form the semisimple part of $\mathsf{MM}(\mathbb{F})$.

[5] There are many other motivic tt-categories (like motivic spectra or étale motives among others) for which the same questions can be asked, with mostly similar short answers.

[6] assuming that the equivalence is compatible with the Betti realization, which we do implicitly

Theorem 5.5 ([Gal21]). *Assume the equivalence* (5.4)*. Then* $\mathrm{Spec}(\mathsf{DM}(\mathbb{F};\mathbb{Q})) = *$ *is a singleton set.*

Proof. Let $\omega\colon \mathsf{MM}(\mathbb{F};\mathbb{Q}) \to \mathbb{Q}\text{-vs}$ be the (faithful, exact, symmetric monoidal) Betti realization, and $\omega\colon \mathsf{D}^{\mathrm{b}}(\mathsf{MM}(\mathbb{F};\mathbb{Q})) \to \mathsf{D}^{\mathrm{b}}(\mathbb{Q})$ also the induced functor on derived categories. We are going to apply Proposition 4.11 to ω (using Exercise 4.13 as well), so let $f\colon \mathbb{1} \to b$ be a morphism in $\mathsf{D}^{\mathrm{b}}(\mathsf{MM}(\mathbb{F}))$ such that $\omega(f) = 0$. The composition $\mathbb{1} \xrightarrow{f} b \to \tau^{\geq 0}b$ vanishes (where $\tau^{\bullet}$ is the cohomological truncation of the complex in the respective (cohomological) degrees). Hence f factors through $\tau^{<0}b$. In other words, we may assume b lives in $\mathsf{D}^{\leq -1}(\mathsf{MM}(\mathbb{F}))$. But then $b^{\otimes n} \in \mathsf{D}^{\leq -n}(\mathsf{MM}(\mathbb{F}))$ and there are no non-zero maps from $\mathbb{1}$ to this subcategory for $n \gg 0$, that is, $f^{\otimes n} = 0$ as required. Indeed, the Tannakian subcategory generated by $b \in \mathsf{MM}(\mathbb{F})$ is equivalent to finite-dimensional representations (over $\mathbb{Q}$) of an algebraic group, and these have finite cohomological dimension in characteristic zero. □

Remark 5.6. Coming back to the question at the beginning of this section, one might see this as a disappointing result. After all, the complexity of motives seems to disappear completely in tt-geometry. But not so fast! The following observation shows that questions about algebraic cycles and the tt-geometry of $\mathsf{DM}(\mathbb{F};\mathbb{Q})$ are in fact connected.

Proposition 5.7 (Ayoub)**.** *Assume* $\mathrm{Spec}(\mathsf{DM}(\mathbb{C};\mathbb{Q})) = *$*. Then Bloch's conjecture on surfaces holds.*

Bloch's conjecture says that for a smooth projective complex surface S of geometric genus 0, the Albanese map

$$\mathrm{CH}_0(S)_0 \to \mathrm{Alb}(S)(\mathbb{C})$$

from the Chow group of zero-cycles of degree 0 is injective. It is known for all surfaces not of general type.

Proof. Our assumption on the spectrum implies that the Betti realization $\mathsf{DM}(\mathbb{C};\mathbb{Q}) \to \mathsf{D}^{\mathrm{b}}(\mathbb{Q})$ is conservative. (It would be enough to know that the kernel (which is necessarily a prime ideal) is the unique closed point of the spectrum.) The claim then follows from [Ayo17, Proposition 2.15]. □

Remark 5.8. The upshot of this discussion is that $\mathrm{Spec}(\mathsf{DM}(\mathbb{F};\mathbb{Q}))$ should be a very simple space but proving this is very difficult.

5.2 Finite coefficients

With finite coefficients there aren't any grand conjectures to guide us, fortunately or unfortunately. But what is certain is that we cannot expect the spectrum to be a singleton. For example, let p be a prime number and recall that

$$\mathrm{Hom}_{\mathsf{DM}(\mathbb{F};\mathbb{Z}/p)}(\mathbb{1}, \mathbb{1}(1)) = \mu_p(\mathbb{F})$$

are the pth roots of unity. Assuming $\mathbb{F}$ contains a primitive one, this defines a non-zero map $\tau\colon \mathbb{1} \to \mathbb{1}(1)$ which under étale realization

$$\mathsf{DM}(\mathbb{F};\mathbb{Z}/p) \to \mathsf{DM}_{\text{ét}}(\mathbb{F};\mathbb{Z}/p) \simeq \mathsf{D}^{\mathrm{b}}(\mathrm{Gal}_{\mathbb{F}};\mathbb{Z}/p)$$

is inverted. It turns out that étale realization is exactly localization at τ (ask the organizers of this conference!). It follows that the kernel is a non-zero prime ideal.

In fact, the literature contains a large number of points of the spectrum and for certain subcategories even complete computations of the space.

Theorem 5.9 ([Vis24]). *The kernel of every isotropic realization* $\mathsf{DM}(\mathbb{F};\mathbb{Z}/p) \to \mathsf{DM}(\tilde{\mathbb{E}}/\tilde{\mathbb{E}};\mathbb{Z}/p)$ *defines a point in* $\mathrm{Spec}(\mathsf{DM}(\mathbb{F};\mathbb{Z}/p))$. *Moreover, such points are equal iff the corresponding field extensions are* p*-equivalent.*

This gives, for many base fields $\mathbb{F}$, an enormous number of points. These are expected to be closed and therefore have no specialization relations among them. We refer to *loc. cit.* and [Vis26] for details.

If one restricts to certain subcategories of motives, complete computations become possible.

Example 5.10. Restricting to the subcategory of Tate motives, $\mathsf{DTM}(\mathbb{F};\mathbb{Z}/p)$, it turns out that in most cases the only prime ideals are 0 and the kernel of étale realization discussed above. That is, $\mathrm{Spec}(\mathsf{DTM}(\mathbb{F};\mathbb{Z}/p))$ is a Sierpinski space. For example, this is true when $\mathbb{F}$ has no real embeddings, e.g. when $\mathbb{F}=\overline{\mathbb{F}}$ [Gal19], or whenever $p\neq 2$. The most convenient way to formulate the result in general is to consider a variant of the comparison map. Consider the bigraded motivic cohomology ring (or "twisted endomorphism ring" of the unit),

$$\mathrm{H}^{\bullet,\bullet'}(\mathbb{F};\mathbb{Z}/p) := \mathrm{Hom}_{\mathsf{DTM}(\mathbb{F};\mathbb{Z}/p)}(\mathbb{1},\mathbb{1}(\bullet')[\bullet]).$$

(If $\mathbb{F}$ contains a primitive pth root of unity, this is nothing but $K^{\mathrm{M}}_{\bullet}(\mathbb{F})/p[\tau]$ with τ in bidegree $(0,1)$ as above.) Then

$$\mathrm{comp}\colon\ \mathrm{Spec}(\mathsf{DTM}(\mathbb{F};\mathbb{Z}/p)) \to \mathrm{Spec}(\mathrm{H}^{\bullet,\bullet'}(\mathbb{F};\mathbb{Z}/p))$$

is a homeomorphism. This has been proved by Schemeil–Sparks (personal communication).

Example 5.11. Beyond Tate motives one may consider the subcategory generated by the motives of smooth projective quadrics. This will now depend heavily on $\mathbb{F}$ as it also reflects the theory of quadratic forms over $\mathbb{F}$. The most interesting case is $p=2$. Schemeil [Sch26] computed the spectrum completely when $\mathbb{F}$ is real closed (for example the real numbers). It is a space of Krull dimension 2 and with countably many points, related to the countable family of Pfister quadrics.

Example 5.12. Another natural subcategory to consider is Artin motives (generated by the motives of 0-dimensional varieties), $\mathsf{DAM}(\mathbb{F};\mathbb{Z}/p)$. Using the Galois–Grothendieck correspondence between 0-dimensional varieties and finite $\mathrm{Gal}_{\mathbb{F}}$-sets this can be translated into a problem in (modular) representation theory. The resulting space $\mathrm{Spec}(\mathsf{DAM}(\mathbb{F};\mathbb{Z}/p))$ turns out to be reasonably complicated [BG25a; BG25b]. For example, its Krull dimension is essentially the maximal number of generators of all subquotients of $\mathrm{Gal}_{\mathbb{F}}$ of p-power order. The closed points are in bijection with conjugacy classes of pro-p-subgroups.

Remark 5.13. These examples give an idea of how complicated $\mathrm{Spec}(\mathsf{DM}(\mathbb{F};\mathbb{Z}/p))$ is likely going to be. Indeed, their inclusion D?M$(\mathbb{F};\mathbb{Z}/p) \to \mathsf{DM}(\mathbb{F};\mathbb{Z}/p)$ satisfies the assumptions of

Proposition 4.11 so that the induced map on spectra,

$$\mathrm{Spec}(\mathsf{DM}(\mathbb{F};\mathbb{Z}/p)) \to \mathrm{Spec}(\mathrm{D?M}(\mathbb{F};\mathbb{Z}/p))$$

is surjective.

This is not an exhaustive list of computations in motivic tt-geometry available in the literature. We invite the reader to explore some other works, such as [Pet13; HO18; Kel20; Gal22; DV25; Spa25].

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